Probability Theory

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Chapter 03: Conditional Probability and Independence

Conditional Probability

Introductory Example

- Suppose that we toss 2 fair dice
- Each of the 36 possible outcomes has the same probability $\frac{1}{36}$
- Suppose further that we observe that the first die is a 3
- Given this information, what is the probability that the sum of the 2 dice equals 8?

- Given that the first die is a 3, the sample space is reduced to
$$\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}$$
- These outcomes should still have equal probabilities.
- The (conditional) probability of each of these outcomes is $\frac{1}{6}$
- The (conditional) probability of each of the other 30 points is 0
- The desired probability is $\frac{1}{6}$

Conditional Probability

Conditional Probability

The conditional probability that an event E occurs given that event F has occurred is
$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$\begin{aligned} P(EF) &= P(F)P(E|F) \\ &= P(E)P(F|E) \end{aligned}$$

Examples

Example 1

A fair coin is flipped twice. What is the conditional probability that both flips land on heads, given that

1. the first flip lands on heads?
2. at least one flip lands on heads?

- Let B be the event that both flips land on heads
- Let F be the event that the first flip lands on heads
- Let A be the event that at least one flip lands on heads

$$\begin{aligned} P(B|F) &= \frac{P(BF)}{P(F)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t)\})} \\ &= \frac{1/4}{1/2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(B|A) &= \frac{P(BA)}{P(A)} \\ &= \frac{P(\{(h,h)\})}{P(\{(h,h),(h,t),(t,h)\})} \\ &= \frac{1/4}{3/4} = \frac{1}{3} \end{aligned}$$

Examples (cont'd)

Example 2

Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an A grade would be $\frac{1}{2}$ in a French course and $\frac{2}{3}$ in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an A in chemistry?

- Let C be the event that Celine takes chemistry
- Let A be the event that she receives an A in whatever course she takes

$$\begin{aligned} P(CA) &= P(C)P(A|C) \\ &= \frac{1}{2} \times \frac{2}{3} = \frac{1}{3} \end{aligned}$$

The Chain Rule

$$P(E_1E_2E_3\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1E_2E_3\cdots E_{n-1})$$

Example

An ordinary deck of 52 playing cards is randomly divided into 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

- $E_1 = \{$A♠ is in any one of the piles$\}$
- $E_2 = \{$A♠ and A♥ are in different piles$\}$
- $E_3 = \{$A♠, A♥, A♦ are all in different piles$\}$
- $E_4 = \{$all 4 aces are in different piles$\}$

$$\begin{aligned} P(E_1E_2E_3E_4) &= P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3) \\ &= 1 \times \frac{39}{51} \times \frac{26}{50} \times \frac{13}{49} \\ &= 0.105 \end{aligned}$$

Bayes's Formula

Let E and F be events. We may express E as $E = EF \cup EF^c$.

$$\begin{aligned} P(E) &= P(EF) + P(EF^c) \\ &= P(E|F)P(F) + P(E|F^c)P(F^c) \\ &= P(E|F)P(F) + P(E|F^c)\big(1 - P(F)\big) \end{aligned}$$

Suppose that $F_1, F_2, \cdots F_n$ are mutually exclusive events such that $S = \bigcup_{i=1}^{n} F_i$, and by writing $E = \bigcup_{i=1}^{n} EF_i$, we obtain

$$\begin{aligned} P(E) &= \sum_{i=1}^{n} P(EF_i) \\ &= \sum_{i=1}^{n} P(E|F_i)P(F_i) \end{aligned}$$

Examples

Example 1

A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. If .5 percent of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

- Let D be the event that the person tested has the disease
- Let E be the event that the test result is positive

$$P(E|D) = .95 \quad P(E|D^c) = .01 \quad P(D) = .005 \quad P(D^c) = .995$$

$$\begin{aligned} P(D|E) &= \frac{P(DE)}{P(E)} \\ &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} \\ &= \frac{(.95)(.005)}{(.95)(.005) + (.01)(.995)} \approx .323 \end{aligned}$$

Examples

Example 1 (cont'd)

... What is the probability that a person has the disease given that the test result is negative?

$$P(E|D) = .95 \quad P(E|D^c) = .01 \quad P(D) = .005 \quad P(D^c) = .995 \quad P(E^c|D^c) = .99 \quad P(E^c|D) = .05$$

$$\begin{aligned} P(D|E^c) &= \frac{P(DE^c)}{P(E^c)} \\ &= \frac{P(E^c|D)P(D)}{P(E^c|D)P(D) + P(E^c|D^c)P(D^c)} \\ &= \frac{(.05)(.005)}{(.05)(.005) + (.99)(.995)} \\ &\approx 0.00025 \end{aligned}$$

Examples (cont'd)

Example 2

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

- $G_1 \triangleq$ event that the first child is a girl ($B_1 \triangleq$ boy)
- $G_2 \triangleq$ event that the second child is a girl ($B_2 \triangleq$ boy)
- $G \triangleq$ event that the child seen with the mother is a girl ($B \triangleq$ boy)

$$P(G_1G_2|G) = \frac{P(G_1G_2G)}{P(G)} = \frac{P(G_1G_2)}{P(G)}$$

$$\begin{aligned} P(G) &= P(G|G_1G_2)P(G_1G_2) + P(G|G_1B_2)P(G_1B_2) \\ &\quad + P(G|B_1G_2)P(B_1G_2) + P(G|B_1B_2)P(B_1B_2) \\ &= P(G_1G_2) + P(G|G_1B_2)P(G_1B_2) + P(G|B_1G_2)P(B_1G_2) \end{aligned}$$

Examples (cont'd)

Example 2 (cont'd)

$$\begin{aligned} P(G_1G_2|G) &= \frac{P(G_1G_2)}{P(G_1G_2) + P(G|G_1B_2)P(G_1B_2) + P(G|B_1G_2)P(B_1G_2)} \\ &= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4}P(G|G_1B_2) + \frac{1}{4}P(G|B_1G_2)} \\ &= \frac{1}{1 + P(G|G_1B_2) + P(G|B_1G_2)} \end{aligned}$$

If we want to assume that the child walking with the mother is the elder child with some probability p

$$P(G_1G_2|G) = \frac{1}{2}$$

If we were to assume that if the children are of different genders, then the mother would choose to walk with the girl with probability q

$$P(G_1G_2|G) = \frac{1}{1 + 2q}$$

Odds of an Event

Definition

The odds of an event A tell how much more likely it is that the event A occurs than it is that it does not occur

$$O(A) = \frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)} \qquad P(A) = \frac{O(A)}{1 + O(A)}$$

if $P(A) = \frac{2}{3}$, then $P(A) = 2P(A^c)$, so the odds are 2 to 1 in favor of A

Consider a hypothesis H that is true with probability $P(H)$, and suppose that new evidence E is introduced

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)} \qquad P(H^c|E) = \frac{P(E|H^c)P(H^c)}{P(E)}$$

$$O(H|E) = O(H) \times \frac{P(E|H)}{P(E|H^c)}$$

Example

Example

An urn contains two type A coins and one type B coin. When a type A coin is flipped, it comes up heads with probability 1/4, whereas when a type B coin is flipped, it comes up heads with probability 3/4. A coin is randomly chosen from the urn and flipped. Given that the flip landed on heads, what is the probability that it was a type A coin?

- Let A be the event that a type A coin was flipped
- Therefore A^c is the event that a type B coin was flipped

$$O(A|\text{heads}) = O(A) \times \frac{P(\text{heads}|A)}{P(\text{heads}|A^c)} = \frac{2/3}{1/3} \times \frac{1/4}{3/4} = \frac{2}{3}$$

$$P(A|\text{heads}) = \frac{O(A|\text{heads})}{1 + O(A|\text{heads})} = \frac{2/3}{1+2/3} = \frac{2}{5}$$

Independent Events

Definition

E is independent of F if knowledge that F has occurred does not change the probability that E occurs.

$$P(E|F) = P(E)$$

Since $P(E|F) = P(EF)/P(F)$, it follows that E is independent of F if

$$P(EF) = P(E)P(F)$$

The fact that the last equation is symmetric in E and F shows that whenever E is independent of F, F is also independent of E.

Two events E and F that are not independent are said to be dependent.

If E and F are independent events, then so are E and F^c.

$$P(E) = P(EF) + P(EF^c) = P(E)P(F) + P(EF^c)$$
$$P(EF^c) = P(E)\big(1 - P(F)\big) = P(E)P(F^c)$$

Example

Example

Suppose that we toss 2 fair dice.

- Let E denote the event that the first die equals 4.
- Let F_1 denote the event that the sum of the dice is 6
- Let F_2 denote the event that the sum of the dice is 7

Are E and F_1 independent? What about E and F_2?

$$P(EF_1) = P(\{(4,2)\}) = \frac{1}{36}$$
$$P(E)P(F_1) = \frac{1}{6} \times \frac{5}{36} = \frac{5}{216} \neq P(EF_1)$$

E and F_1 are not independent

$$P(EF_2) = P(\{(4,3)\}) = \frac{1}{36}$$
$$P(E)P(F_2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = P(EF_2)$$

E and F_2 are independent

Independence of a Set of Events

Three events E, F, and G are said to be independent if

$$P(EFG) = P(E)P(F)P(G)$$
$$P(EF) = P(E)P(F)$$
$$P(EG) = P(E)P(G)$$
$$P(FG) = P(F)P(G)$$

The events $E_1, E_2, \cdots, E_n$ are said to be independent if, for every subset $E_{1'}, E_{2'}, \cdots, E_{r'}$, $r' \leq n$ of these events

$$P(E_{1'}, E_{2'}, \cdots, E_{r'}) = P(E_{1'})P(E_{2'})\cdots P(E_{r'})$$

An infinite set of events are defined to be independent if every finite subset of those events is independent.

Examples

Example

Independent trials consisting of rolling a pair of fair dice are performed. What is the probability that an outcome of 5 appears before an outcome of 7 when the outcome of a roll is the sum of the dice?

$E_n \triangleq$ event that no 5 or 7 appears on the first $n-1$ trials and a 5 appears on the n^{th} trial

Since $P\{5 \text{ on any trial}\} = \frac{4}{36}$ and $P\{7 \text{ on any trial}\} = \frac{6}{36}$, we obtain, by the independence of trials

$$P(E_n) = \left(1 - \frac{10}{36}\right)^{n-1} \times \frac{4}{36} = \frac{1}{9}\left(\frac{13}{18}\right)^{n-1}$$

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) = \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{13}{18}\right)^{n-1} = \frac{1}{9} \times \frac{1}{1-\frac{13}{18}} = \frac{2}{5}$$

$P(.|F)$ is a Probability

Conditional probabilities satisfy all of the properties of ordinary probabilities

1. $0 \leq P(E|F) \leq 1$
2. $P(S|F) = 1$
3. For any sequence of mutually exclusive events $E_1, E_2, \cdots$,

$$P\left(\bigcup_{i=1}^{\infty} E_i|F\right) = \sum_{i=1}^{\infty} P(E_i|F)$$

If we define $Q(E) \triangleq P(E|F)$, then $Q(E)$ may be regarded as a probability function on the events of S. Hence, all of the propositions previously proved for probabilities apply to $Q(E)$. For instance,

$$Q(E_1 \cup E_2) = Q(E_1) + Q(E_2) - Q(E_1E_2)$$
$$P(E_1 \cup E_2|F) = P(E_1|F) + P(E_2|F) - P(E_1E_2|F)$$

$P(.|F)$ is a Probability (cont'd)

$$Q(E_1E_2E_3\cdots E_n) = Q(E_1)Q(E_2|E_1)Q(E_3|E_1E_2)\cdots Q(E_n|E_1E_2E_3\cdots E_{n-1})$$
$$P(E_1E_2E_3\cdots E_n|F) = P(E_1|F)P(E_2|E_1F)P(E_3|E_1E_2F)\cdots P(E_n|E_1E_2E_3\cdots E_{n-1}F)$$

If we define $Q(E_1|E_2) \triangleq Q(E_1E_2)/Q(E_2)$, then

$$Q(E_1) = Q(E_1|E_2)Q(E_2) + Q(E_1|E_2^c)Q(E_2^c)$$
$$P(E_1|F) = P(E_1|E_2F)P(E_2|F) + P(E_1|E_2^cF)P(E_2^c|F)$$

We say that the events E_1 and E_2 are *conditionally independent* given F if, given that F occurs, the conditional probability that E_1 occurs is unchanged by information as to whether or not E_2 occurs

$$P(E_1|E_2F) = P(E_1|F)$$

or, equivalently,

$$P(E_1E_2|F) = P(E_1|F)P(E_2|F)$$

Example

Example

A female chimp has given birth. It is not certain, however, which of two male chimps is the father. Before any genetic analysis has been performed, it is felt that the probability that male #1 is the father is p and the probability that male #2 is the father is $1-p$. DNA obtained from the mother, male #1, and male #2 indicate that, on one specific location of the genome, the mother has the gene pair (A,A), male #1 has the gene pair (a,a), and male #2 has the gene pair (A,a). If a DNA test shows that the baby chimp has the gene pair (A,a), what is the probability that male #1 is the father?

Let all probabilities be conditional on the event that the mother has the gene pair (A,A), male #1 has the gene pair (a,a), and male #2 has the gene pair (A,a).

Example (cont'd)

- Let M_i be the event that male $\#i$, $i=1,2$, is the father
- Let $B_{A,a}$ be the event that the baby chimp has the gene pair (A,a)

$$\begin{aligned} P(M_1|B_{A,a}) &= \frac{P(M_1B_{A,a})}{P(B_{A,a})} \\ &= \frac{P(B_{A,a}|M_1)P(M_1)}{P(B_{A,a}|M_1)P(M_1)+P(B_{A,a}|M_2)P(M_2)} \\ &= \frac{1\times p}{1\times p + 1/2\times(1-p)} \\ &= \frac{2p}{1+p} > p \qquad (p<1) \end{aligned}$$

The information that the baby's gene pair is (A,a) increases the probability that male #1 is the father.